

Solution to the Balitsky-Kovchegov equation in the saturation domain

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Abstract

The solution to the Balitsky-Kovchegov equation is found in the deep saturation domain. The controversy between different approaches regarding the asymptotic behaviour of the scattering amplitude is solved. It is shown that the dipole amplitude behaves as $1 - \exp(-z + \ln z)$ with $z = \ln(r^2 Q_s^2)$ (r -size of the dipole, Q_s is the saturation scale) in the deep saturation region. This solution is developed from the scaling solution to the homogeneous Balitsky-Kovchegov equation. The dangers associated with making simplifications in the BFKL kernel, to investigate the asymptotic behaviour of the scattering amplitude, is pointed out. In particular, the fact that the Balitsky-Kovchegov equation belongs to the Fisher-Kolmogorov-Petrovsky-Piscounov -type of equation, needs further careful investigation.

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1 Introduction

The main objective of this paper is to find the solution to the non-linear Balitsky-Kovchegov equation [1, 2] in the saturation domain. There exist two solutions for dipole scattering amplitude N which were found in Refs. [3, 4, 5]. They can be presented in the form:

$$N(Y, r; b) = 1 - e^{-\phi(z)} \quad (1.1)$$

where

$$z = \ln(Q_s^2(Y; b) r^2) = \alpha_S C Y + \ln(r^2/r_0^2(b)) \quad (1.2)$$

and Q_s is a saturation scale [6, 7, 8, 9]. Constant C in Eq. (1.2) is defined as

$$C = \frac{\chi(\gamma_{cr})}{1 - \gamma_{cr}} \quad (1.3)$$

and χ is the BFKL kernel [10]

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1 - \gamma) \quad (1.4)$$

where $\psi(\gamma) = d \ln \Gamma(\gamma) / d\gamma$ and $\Gamma(\gamma)$ is the Euler gamma function.

The value for the critical anomalous dimension γ_{cr} is determined by the equation [6, 9, 11]

$$\frac{\chi(\gamma_{cr})}{1 - \gamma_{cr}} = - \frac{d\chi(\gamma_{cr})}{d\gamma_{cr}}. \quad (1.5)$$

In the first solution [3]

$$\phi(z) = \frac{z^2}{2C} \quad (1.6)$$

while the second one has the form [4, 5]

$$\phi(z) = z - \ln(z). \quad (1.7)$$

These two solutions lead to quite different approaches to the saturation boundary $N \rightarrow 1$. However, since the equation is a non-linear one, we cannot claim that only Eq. (1.7) survives at high energies.

Eq. (1.6) is well accepted by the experts, while Eq. (1.7) is still considered by the experts as shaky, mostly because its derivation has not reached a stage of transparency as the first solution.

In this paper we present (i) the derivation of both solutions in the framework of the same method; (ii) a simple explanation of both solutions; and (iii) the general form of approaching the unitarity limit.

2 Solution to the Balitsky-Kovchegov equation (general approach)

2.1 Equation

The Balitsky-Kovchegov equation [1, 2] which we solve in this section, has a form:

$$\frac{\partial N(r, Y; b)}{\partial Y} = \frac{C_F \alpha_S}{\pi^2} \int \frac{d^2 r' r^2}{(\vec{r} - \vec{r}')^2 r'^2} \quad (2.1)$$

$$\left(2N\left(r', Y; \vec{b} + \frac{1}{2}(\vec{r} - \vec{r}')\right) - N(r, Y; b) - N\left(r', Y; \vec{b} - \frac{1}{2}(\vec{r} - \vec{r}')\right) N\left(\vec{r} - \vec{r}', Y; \vec{b} - \frac{1}{2}\vec{r}'\right) \right)$$

where $N(r, Y; b)$ is the scattering amplitude of interaction for the dipole with the size r and rapidity $Y = \ln(1/x)$ (x is the Bjorken variable), at impact parameter b .

It is useful to consider the non-linear equation in a mixed representation, fixing the impact parameter b , and introducing the transverse momenta as conjugate variable to the dipole sizes. The relations between these two representations are given by the following equations

$$N(r, y; b) = r^2 \int_0^\infty k dk J_0(k r) \tilde{N}(k, y; b); \quad (2.2)$$

$$\tilde{N}(k, y; b) = \int_0^\infty \frac{dr}{r} J_0(k r) N(r, y; b); \quad (2.3)$$

In this representation the non-linear equation reduces to the form [6, 13, 4]

$$\frac{\partial \tilde{N}(k, y; b)}{\partial y} = \bar{\alpha}_S \left(\chi(\hat{\gamma}(\xi)) \tilde{N}(k, y; b) - \tilde{N}^2(k, y; b) \right) \quad (2.4)$$

and $\chi(\hat{\gamma}(\xi))$ is an operator defined as

$$\hat{\gamma}(\xi) = 1 + \frac{\partial}{\partial \xi} \quad (2.5)$$

where $\xi = \ln(k^2 R^2)$, and k is the conjugate variable to the colour dipole size and R is the size of the target. In this definition of the variable ξ we implicitly assume that $b \ll R$, and the amplitude N does not depend on b . The alternative approach for large values of the impact parameter was developed in Ref. [4], where the definition of variable ξ is quite different.

2.2 Goal and assumptions

Our main goal is to find the solution in the saturation region where $r^2 Q_s^2 \gg 1$. Due to s -channel unitarity constraint [14]¹ the scattering amplitude in space representation (N) should

¹See also Ref. [15] for an application of the Froissart boundary for hard processes

be less than 1. In the momentum representation this limit means that $\tilde{N} \rightarrow \frac{1}{2}\xi$. We can obtain this estimate just using Eq. (2.3), and noticing that for $r > 1/k$ the integral over r is small due to oscillating behaviour of J_0 in Eq. (2.3). Therefore, for small k Eq. (2.3) reduces to

$$\tilde{N}(k, y; b) = \int_0^{\frac{1}{k}} \frac{dr}{r} N(r, y; b) = \frac{1}{2} \int_\xi^\infty d \ln(1/r^2) N(r, y; b) \quad (2.6)$$

Assuming that $N=1$ for all $r > 1/Q_s$ ($\ln(1/r) < \ln(Q_s)$) we obtain from Eq. (2.6) that

$$\tilde{N}(k, y; b) \rightarrow \frac{1}{2} \ln(Q_s^2/k^2) \quad (2.7)$$

Therefore we know the asymptotic behaviour of the amplitude, and we need to find how our function approaches Eq. (2.7).

In the saturation region we expect the so called geometrical scaling behaviour of the scattering amplitude which was proven in Ref. [9], for such equations (see also Ref. [13] for more general arguments and a more rigorous proof, and Ref. [12] for an observation that geometrical scaling behaviour could be correct, even in the part of perturbative QCD kinematic region). It means that $\tilde{N}(k, y; b)$ is a function of the single variable

$$\hat{z} = \ln(Q_s^2(y, b)/k^2) = \bar{\alpha}_S \frac{\chi(\gamma_{cr})}{1 - \gamma_{cr}} (Y - Y_0) - \xi - \beta(b); \quad (2.8)$$

where γ_{cr} is a solution of Eq. (1.5) [6, 11, 13] and Y_0 is the initial rapidity. Function $\beta(b)$ depends on impact parameter, but we will not discuss it here.

Introducing a function $\phi(z)$ we are looking for the solution of the equation in the form

$$\tilde{N}(\hat{z}) = \frac{1}{2} \int^{\hat{z}} dz' (1 - e^{-\phi(z')}) ; \quad (2.9)$$

Eq. (2.9) includes the geometrical scaling behaviour and leads to the asymptotic behaviour of Eq. (2.7).

Our assumption that function ϕ is a smooth function, such that $\phi_{zz} \ll \phi_z \phi_z$ where we denote $\phi_z = d\phi/dz$ and $\phi_{zz} = d^2\phi/(dz)^2$ is essential.. This property allows us to rewrite

$$\frac{d^n}{(dz)^n} e^{-\phi(z)} = (-\phi_z)^n e^{-\phi(z)} \quad (2.10)$$

Eq. (2.10) means that we can use the semi-classical approach for the solution to Eq. (2.1) [4].

2.3 Reduction of the Balitsky-Kovchegov equation to the equation in one variable in the saturation domain

Substituting in Eq. (2.4) \tilde{N} in the form of Eq. (2.9), and replacing Y by \hat{z} we obtain

$$\bar{\alpha}_S \frac{\chi(\gamma_{cr})}{1 - \gamma_{cr}} \frac{d\tilde{N}(\hat{z})}{d\hat{z}} = \bar{\alpha}_S \left(\chi(1 - f) \tilde{N}(\hat{z}) - \tilde{N}^2(\hat{z}) \right) \quad (2.11)$$

where f denotes $f = d/d\hat{z} = -\partial/\partial\xi$ in Eq. (2.11).

Differentiating both part of Eq. (2.11) with respect to z we reduce Eq. (2.11) to the form

$$\begin{aligned} \frac{1}{2}e^{-\phi(\hat{z})} \phi'_z(\hat{z}) &= f\chi(1-f)\hat{N}(\hat{z}) - \hat{N}(\hat{z})(1 - e^{-\phi(z)}) \\ &= \frac{1}{2}(f\chi(1-f) - 1)\hat{N}(\hat{z}) + \hat{N}(\hat{z})e^{-\phi(z)} \end{aligned} \quad (2.12)$$

An important property of function $f\chi(1-f) - 1$, is the fact that at small f it has an expansion that starts ² from f^3 . Since in our case f is operator $f \equiv \frac{d}{dz}$, it means that the operator $f\chi(1-f) - 1$ contains the third and higher derivatives with respect to z . Therefore, one can see that the first term on r.h.s. of Eq. (2.12) is proportional to $e^{-\phi(\hat{z})}$ (see Eq. (2.10)). Canceling $e^{-\phi}$ on both sides of Eq. (2.12), and once more taking the derivative with respect to \hat{z} we reduce Eq. (2.12) to the form:

$$\frac{\chi(\gamma_{cr})}{1 - \gamma_{cr}} \frac{d^2 \phi}{(d\hat{z})^2} = \left(1 - e^{-\phi(\hat{z})}\right) - \frac{dL(\phi_z)}{d\phi_z} \frac{d^2 \phi}{(d\hat{z})^2}; \quad (2.13)$$

$$L(\phi_z) = \frac{\phi_z \chi(1 - \phi_z) - 1}{\phi_z}; \quad (2.14)$$

Function $dL/d\phi'_z$ decreases at large values of the argument but has double pole singularities in all integer points ($\phi'_z = 1, 2, 3, \dots$).

2.4 Two solutions.

The existence of two solutions with sufficiently different forms of the dipole amplitude approaching its asymptotic value ($N = 1$ in space representation and $\tilde{N} = \frac{1}{2} \ln(Q_s^2/k^2)$ in the momentum representation) can be seen directly from Eq. (2.13). Indeed, we expect that $\phi(z)$ is large at large values of z and, therefore, we can neglect the term $e^{-\phi(\hat{z})}$ in Eq. (2.13). The first solution can be obtained from Eq. (2.13) assuming that $dL/d\phi'_z$ gives a small contribution while $\phi'(z)$ is large. If it is so Eq. (2.13) reduces to the simple equation

$$\frac{\chi(\gamma_{cr})}{1 - \gamma_{cr}} \frac{d^2 \phi}{(d\hat{z})^2} = 1 \quad (2.15)$$

which leads to

$$\phi(\hat{z}) = \frac{1 - \gamma_{cr}}{\chi(\gamma_{cr})} \frac{\hat{z}^2}{2} \quad (2.16)$$

²It should be stressed that the simplified model for $\chi(1-f) = \frac{1}{f(1-f)}$ does not have this property. This is an explanation why in Ref. [3] where this model was used, the solution was missed as we will discuss below.

at large \hat{z} . The clearest case when we, indeed have this solution, is the simplification of the BFKL kernel which was considered in Ref. [3]. Namely, the kernel was taken as

$$\omega(\gamma) = \frac{\alpha_S N_c}{\pi} \begin{cases} \frac{1}{\gamma} & \text{for } r^2 Q_s^2 < 1; \\ \frac{1}{1-\gamma} & \text{for } r^2 Q_s^2 > 1; \end{cases} \quad (2.17)$$

instead of the full BFKL kernel $\omega(\gamma) = \frac{\alpha_S N_c}{\pi} \chi(\gamma)$.

In this model function L is equal to zero and Eq. (2.16) gives the only solution in the saturation domain.

For the full BFKL kernel there exists another possibility to have a solution: the term with $dL/d\phi'_z$ compensates 1 in the r.h.s. of Eq. (2.13), and the l.h.s. is still small.

Function $\frac{dL(\phi_z)}{d\phi_z}$ being small for $\phi_z < 1$ has a singularity at $\phi \rightarrow 1$ namely

$$\frac{dL(\phi_z)}{d\phi_z} \longrightarrow \frac{1}{(1 - \phi_z)^2} \quad \text{for} \quad \phi_z \longrightarrow 1$$

Therefore, the first requirement leads to the equation

$$\frac{1}{(1 - \phi_z)^2} \frac{d^2 \phi}{(d\hat{z})^2} = 1. \quad (2.18)$$

For large \hat{z} Eq. (2.18) has a solution

$$\phi(\hat{z}) = \hat{z} - \ln \hat{z} \quad (2.19)$$

which can be verified by explicit calculations. It should be stressed that $\phi(\hat{z})$ of Eq. (2.19) satisfies all conditions of a smooth function that has been used for the derivation of Eq. (2.13). Eq. (2.10) holds since $\phi_{zz} \approx 1/z^2 \ll \phi_z^2 \approx 1$. We can also check that the l.h.s. of Eq. (2.13) is proportional to $1/z^2$ and it can be neglected.

We check the validity of the solution of Eq. (2.19) in more direct way searching for the correction to Eq. (2.19) due to a violation of Eq. (2.10). The need to do this, arises from the appearance of a contribution of the order of $1/z^2$ in ϕ_z^2 which we cannot guarantee. Searching for such corrections we replace Eq. (2.10) by a new equation

$$\frac{d^n}{(dz)^n} e^{-\phi(z)} = \left((-\phi_z)^n - n(-\phi_z)^{n-1} \phi_{zz} \right) e^{-\phi(z)} \quad (2.20)$$

Using Eq. (2.20) we obtain

$$\frac{\chi(\gamma_{cr})}{1 - \gamma_{cr}} \frac{d^2 \phi}{(d\hat{z})^2} = \left(1 - e^{-\phi(\hat{z})} \right) - \frac{dL(\phi_z)}{d\phi_z} \frac{d^2 \phi}{(d\hat{z})^2} + \frac{d^2 L(\phi_z)}{(d\phi_z)^2} \left(\frac{d^2 \phi}{(d\hat{z})^2} \right)^2 \quad (2.21)$$

instead of Eq. (2.13). Eq. (2.18) has the following form

$$\frac{1}{(1 - \phi_z)^2} \frac{d^2 \phi}{(d\hat{z})^2} = 1 + \frac{2}{(1 - \phi_{0,z})^3} \left(\frac{d^2 \phi_0}{(d\hat{z})^2} \right)^2 \quad (2.22)$$

where we denote by ϕ_0 the solution of Eq. (2.19). Substituting $\phi_z = \phi_{0,z} + \Delta\phi_z$ in Eq. (2.22) we obtain that

$$\frac{d\Delta\phi_z}{d\hat{z}} = \frac{2}{(1 - \phi_{0,z})} \left(\frac{d^2 \phi_0}{(d\hat{z})^2} \right)^2 \rightarrow \frac{2}{\hat{z}^3} \quad (2.23)$$

One can see that Eq. (2.23) leads to $\Delta\phi_z \propto 1/\hat{z}^2$ and the solution is

$$\phi(\hat{z}) = \hat{z} - \ln \hat{z} + \frac{1}{\hat{z}} = \phi_0(\hat{z}) + O(1/\hat{z}) \quad (2.24)$$

Hence, Eq. (2.19) is a solution.

It should be stressed that Eq. (2.19) is a solution to the homogeneous equation (Eq. (2.13) with the l.h.s. equal to zero). Therefore, the dependence on Y in this solution stems only from the matching of this solution to the linear equation at $\hat{z} < 0$. As has been shown (see [6, 11, 12]) the solution of the linear equation behaves as

$$N(\hat{z}) = N_0 \exp((1 - \gamma_{cr})\hat{z}) \quad (2.25)$$

We will discuss this matching in more details later.

2.5 The complete solution and matching with the pQCD domain

Assuming that in Eq. (2.13), $\phi'_z(\hat{z})$ is a function of $\phi(\hat{z})$ we can rewrite this equation in the form

$$\left(\frac{\chi(\gamma_{cr})}{1 - \gamma_{cr}} + \frac{dL(\phi'_z)}{d\phi'_z} \right) \phi'_z(\hat{z}) \frac{d\phi'_z(\phi)}{d\phi} = 1 - e^{-\phi(\hat{z})} \quad (2.26)$$

Integrating Eq. (2.26) with respect to ϕ we reduce this equation to the form which gives the implicit solution for \hat{z} as a function of ϕ .

$$\begin{aligned} \frac{\chi(\gamma_{cr})}{1 - \gamma_{cr}} \frac{1}{2} (\phi'_z - \phi'_{z,0}) + \chi(1 - \phi'_z) \phi'_z - \chi(1 - \gamma_{cr}) (1 - \gamma_{cr}) - \int_{1-\gamma_{cr}}^{\phi'_z} \chi(\tilde{\phi}'_z) d\tilde{\phi}'_z + \ln(\phi'_z/\phi'_{z,0}) = \\ = \phi + e^{-\phi} - \phi_0 - e^{\phi_0} \end{aligned} \quad (2.27)$$

where $\phi'_{z,0}$ and ϕ_0 are the initial conditions, namely, the value of function ϕ and ϕ'_z at $\hat{z} = 0$. There are a number of relations between them. The first one comes from the matching of the logarithmic derivatives at $\hat{z} = 0$:

$$\frac{\frac{1}{2}(1 - e^{-\phi_0})}{N_0} = 1 - \gamma_{cr} \quad (2.28)$$

The second relation originates from the matching of the logarithmic derivatives for dN/dz using the fact that the function N has the following form near to the saturation line: $N(z) = N_0 e^{(1-\gamma_{cr})z}$ (see Eq. (2.25)), and that the geometrical scaling behaviour works even for negative z [12]. This relation is

$$\phi'_{z,0} e^{-\phi_0} = (1 - \gamma_{cr}) (1 - e^{-\phi_0}) . \quad (2.29)$$

Unfortunately, with the initial conditions given by Eq. (2.28) and Eq. (2.29), we could not solve Eq. (2.27) analytically. Fig. 2.5 presents the numerical solution of this equation for positive z as a function of the initial condition, namely, N_0 (see Eq. (2.28) and Eq. (2.29)).

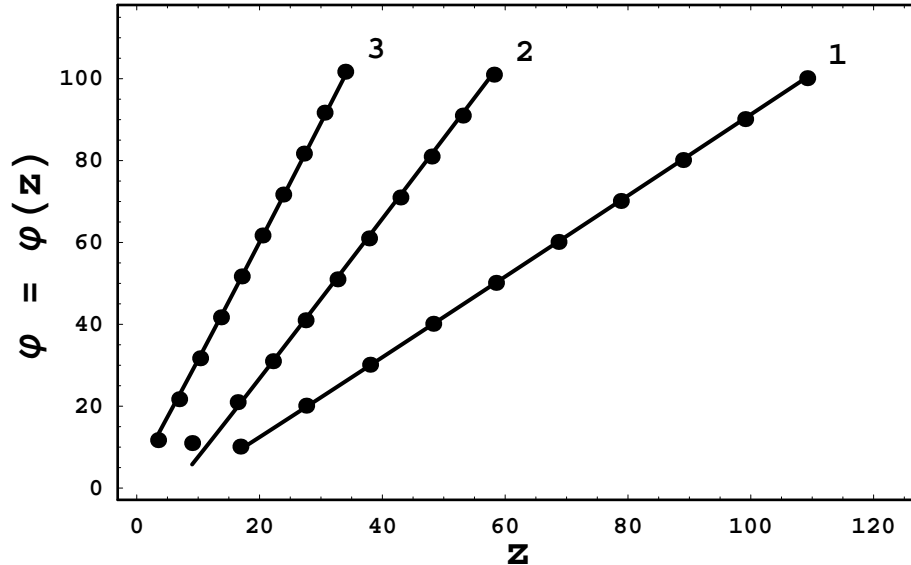


Figure 1: *The numerical solution to Eq. (2.26) as a function of the initial condition N_0 . Curves 1, 2 and 3 correspond to $N_0 = 0.1, 0.5, 0.65$, respectively. The lines are the asymptotic solution $\phi(z) = n z - \ln z + \ln \ln z$ with $n = 1, 2, 3$ for curves 1, 2, and 3, respectively.*

One can see that the type of asymptotic behaviour depends on the value of N_0 at $z = 0$. It has a simple explanation since $dL(\phi_z)d\phi_z \rightarrow (n - \phi_z)^{-2}$. If ϕ_z at $z = 0$ is smaller than unity we have the solution given by Eq. (2.19). However, if $2 > \phi_z > 1$ at $z = 0$, we have $\phi_z(z) \rightarrow 2$ at large z . For $3 > \phi_z > 2$ the asymptotic behaviour is $3z - \ln z$.

It should be stressed that we cannot trust Eq. (2.27) for $z \rightarrow 0$ since we cannot justify Eq.(2.10) in this region. In vicinity of $z = 0$ this equation holds for the function $N(z)$ rather than for the function $\phi(z)$. Nevertheless if $N_0 \ll 1$ ϕ_0 is small as well and our approach could be justified. From Fig. 2.5 one sees that at small N_0 we have the asymptotic behaviour given by Eq. (2.19).

3 Linearized equation in the saturation domain

The key problem is the fact that the master equation is a non-linear one. Therefore, generally speaking, we cannot conclude that the asymptotic behaviour of the solution can be written in the form (in space representation)

$$N(z) = 1 - e^{-\phi_1} - e^{-\phi_2} \quad (3.1)$$

where ϕ_1 is the solution of Eq. (2.16), while ϕ_2 is given by Eq. (2.19).

However, the form of Eq. (2.9) shows us that the corrections are small in the region of large z and, we can therefore try to find the linear equation in this region for the function $e^{-\phi}$. Denoting $e^{-\phi}$ by $S(r, Y; b)$ we obtain the following equation for S from Eq. (2.1):

$$-\frac{\partial S(r, Y; b)}{\partial Y} = \frac{C_F \alpha_S}{\pi^2} \int \frac{d^2 r' r^2}{(\vec{r} - \vec{r}')^2 r'^2} \left(S(r, Y; \vec{b}) - S(r', Y; \vec{b} - \frac{1}{2}(\vec{r} - \vec{r}')) S(\vec{r} - \vec{r}', Y; \vec{b} - \frac{1}{2}\vec{r}') \right) \quad (3.2)$$

The first solution comes from the region of integration $r' \sim r \gg 1/Q_s$ for the second term on l.h.s. of the equation. In this region $S \ll 1$, and therefore, the non-linear term in Eq. (3.2) can be neglected. The linear equation has a very simple form

$$\frac{\partial S(r, Y; b)}{\partial Y} = -\frac{C_F \alpha_S}{\pi} \int_{1/Q_s^2}^{r^2} \frac{dr'^2 r^2}{|r^2 - r'^2| r'^2} S(r, Y; \vec{b}) = -\bar{\alpha}_S \ln(Q_s^2 r^2) S(r, Y; \vec{b}) \quad (3.3)$$

Substituting a new variable z (see Eq. (1.2)) Eq. (3.3) has the form:

$$\frac{\chi(1 - \gamma_{cr})}{1 - \gamma_{cr}} \frac{dS(z)}{dz} = -z S(z) \quad (3.4)$$

and solution to Eq. (3.4) is very simple, namely,

$$S(z) = \exp\left(-\frac{1 - \gamma_{cr}}{\chi(1 - \gamma_{cr})} \frac{z^2}{2}\right) \quad (3.5)$$

The simple derivation of this solution makes it transparent.

The second solution comes from quite a different region in integration in Eq. (3.2), namely, r' or $r - r'$ are much smaller than r and, basically, of the order of $1/Q_s$ since our solution depends on z . Let us assume for the sake of presentation that $1/Q_s \approx |\vec{r} - \vec{r}'| \ll r$. For such small distances we can replace $S(\vec{r} - \vec{r}', Y, b)$ by 1 ($S(\vec{r} - \vec{r}', Y, b) = 1$). Indeed, in pQCD region $1 - S$ is the scattering amplitude and this amplitude is small. Therefore, the linear equation that governs the asymptotic behaviour of S has the following form:

$$-\frac{\partial S(r, Y)}{\partial Y} = \frac{C_F \alpha_S}{\pi^2} \int \frac{d^2 r' r^2}{(\vec{r} - \vec{r}')^2 r'^2} \left(S(r, Y) - 2 S(\vec{r}', Y)_{|\vec{r} - \vec{r}'| \approx 1/Q_s \ll r} \right) \quad (3.6)$$

In Eq. (3.6) we neglected the impact parameter dependence of S .

One can recognize in Eq. (3.6) the BFKL equation, but with an overall sign minus in front of the r.h.s., and with the restriction that the second term is valid only if $|\vec{r} - \vec{r}'| \approx 1/Q_s \ll r$. The factor 2 in front of the second term on the r.h.s. of Eq. (3.6), arise from the fact that we have the same contribution from the kinematic region where $r' \approx 1/Q_s \ll r$. We can rewrite Eq. (3.6) in the form which is even closer to the BFKL equation by replacing the second term on r.h.s. of Eq. (3.6) by

$$\int \frac{d^2 r' r^2}{(\vec{r} - \vec{r}')^2 r'^2} S(\vec{r}', Y)_{|\vec{r} - \vec{r}'| \approx 1/Q_s \ll r} \rightarrow \int \frac{d^2 r' r^2}{(\vec{r} - \vec{r}')^2 r'^2} \left(S(\vec{r}', Y) - \frac{1}{2} S(r, Y) \right) \quad (3.7)$$

In Eq. (3.7) we subtracted the region of integration $1/Q_s \ll r' \ll r$ or/and $1/Q_s \ll r' \gg r$. Indeed, rewriting the kernel in Eq. (3.7) for $r' < r$ in the form (after integrating over azimuthal angle)

$$\int \frac{d^2 r' r^2}{(\vec{r} - \vec{r}')^2 r'^2} \rightarrow \pi \int^{r^2} d r'^2 \left(\frac{1}{r^2 - r'^2} + \frac{1}{r'^2} \right) \quad (3.8)$$

one can see that the first term has been taken into account in Eq. (3.6) since the region $r' \approx r$ gives the dominant contribution to this term. In this statement we assume implicitly that $S(r', Y)$ is steeply decreasing in the saturation region. We have to subtract the second term. We can replace r' by r in S since this function is large only at $r' = r$. Returning to the full kernel we obtain Eq. (3.8).

Taking Eq. (3.8) into account we can rewrite Eq. (3.6) in the form

$$-\frac{\partial S(r, Y)}{\partial Y} = \frac{C_F \alpha_S}{\pi^2} \int \frac{d^2 r' r^2}{(\vec{r} - \vec{r}')^2 r'^2} (S(r, Y) + \{S(r, Y) - 2 S(\vec{r}', Y)\}_{BFKL}) \quad (3.9)$$

The second term on the r.h.s. of Eq. (3.9) is the kernel of the BFKL equation.

Eq. (3.9) is very similar to the BFKL equation, but it has a negative sign in front and also an additional term, proportional to $S(r, Y)$.

Using the form

$$S(r', Y) = S(z) e^{-\phi'_z \ln(r'^2/r^2)} \quad (3.10)$$

we can see that Eq. (3.9) can be reduced to Eq. (2.13). However, we do not need to take into account all the terms in Eq. (2.13). The main contribution comes from the first term on the r.h.s. of Eq. (3.9) and from the second term in the region of integration $r' > r$. Indeed, substituting Eq. (3.10) in the second term on the r.h.s. of Eq. (3.9) the relevant contribution appears as

$$\begin{aligned} \int_{r^2}^{\infty} \frac{r^2 dr'^2}{(r'^2 - r^2) r'^2} \left(\left(\frac{r'^2}{r^2} \right)^{-\phi'_z} - 1 \right) = \\ \int_0^1 \frac{dt}{1-t} (t^{\phi'_z} - 1) = \psi(1) - \psi(1 + \phi'_z) \rightarrow \frac{1}{1 - \phi'_z} \end{aligned} \quad (3.11)$$

Using the variable z instead of Y in Eq. (3.9), we obtain the simple equation which contains both discussed solutions:

$$\frac{\chi(1-\gamma_{cr})}{1-\gamma_{cr}} \frac{dS(z)}{dz} = z S(z) + \frac{1}{1-\phi'_z} S(z) \quad (3.12)$$

Using $S(z) = e^{-\phi(z)}$ we can easily reduce Eq. (3.12) to an algebraic equation for ϕ'_z , namely,

$$\frac{\chi(\gamma_{cr})}{1-\gamma_{cr}} \phi'_z(z) = -z + \frac{1}{1-\phi'_z} \quad (3.13)$$

with the solution

$$\phi'_z(\pm) = \frac{1}{2C} \left(z + C \pm \sqrt{(z+C)^2 + 4C(1-z)} \right) \quad (3.14)$$

where C is given by Eq. (1.3). At large z

$$\phi'_z^{(+)} \rightarrow \frac{z}{C} \quad (3.15)$$

$$\phi'_z^{(-)} \rightarrow 1 - \frac{1}{z} \quad (3.16)$$

The second term on the r.h.s. of Eq. (3.12) is very small if we substitute the solution $\phi'_z^{(+)}$ of Eq. (3.15) into Eq. (3.12). Therefore, this branch leads to

$$S_1(z) \equiv e^{-\phi_1} = e^{-\int^z dz' \phi'_z^{(+)}(z')} = \exp\left(-\frac{z^2}{2C}\right) \quad (3.17)$$

The branch $\phi'_z^{(-)}$ leads a small l.h.s. of Eq. (3.12) and therefore, the solution leads to as a cancellation of the first and second terms on the r.h.s. of Eq. (3.12). The solution takes the form

$$S_2(z) \equiv e^{-\phi_2} = e^{-\int^z dz' \phi'_z^{(-)}(z')} = z \exp(-z) \quad (3.18)$$

The solution $S_2(z)$ is the solution to the homogeneous equation (see Eq. (3.9) with $\partial S(r, Y)/\partial Y = 0$). The entire dependence of this solution on energy (Y) stems from the matching with the solution of the linear equation for negative z .

4 General solution for approaching the unitarity boundary

Here, we are going to find a general solution to Eq. (3.9). We can solve this equation which is an equation of the BFKL -type, using Mellin transform which we use for solution of the BFKL equation:

$$S(z) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} df s(f) e^{fz} \quad (4.1)$$

The main observation is that the function $(r'^2/r^2)^f$ is an eigenfunction of the BFKL kernel [10]. As we have seen (see Eq. (3.11)) the integration in the region $r, > r$ leads to the eigenvalue

$$\int_0^1 \frac{dt}{1-t} (t^{-f} - 1) = \psi(1) - \psi(1-f) \quad (4.2)$$

For $r' < r$ we have the following integral after integration over azimuthal angle :

$$\begin{aligned} \int_0^{r^2} \frac{r^2 dr'^2}{(r^2 - r'^2) r'^2} \left(\left(\frac{r'^2}{r^2} \right)^f - 1 \right) &= - \int_{1/Q_s^2}^{r^2} \frac{dr'^2}{r'^2} + \int_0^{r^2} \frac{dr'^2}{r^2 - r'^2} \left(\left(\frac{r'^2}{r^2} \right)^{f-1} - 1 \right) = \\ &= -z + \int_0^1 \frac{dt}{1-t} (t^{f-1} - 1) = -z + (\psi(1) - \psi(f)) \end{aligned} \quad (4.3)$$

Collecting Eq. (4.2) and Eq. (4.3) we reduce Eq. (3.9) to the form:

$$C f s(f) = \frac{ds(f)}{df} + \chi(f) s(f) \quad (4.4)$$

To obtain Eq. (4.4) we used the result that a multiplication by z translates into operator $-d/df$ for the Mellin transform.

Eq. (4.4) has the following solution

$$s(f) = A e^{X(f)} \quad (4.5)$$

where

$$X(f) = \int_0^f df' (C f' - \chi(f')) = C \frac{f^2}{2} - 2\psi(1)f - \ln \Gamma(1-f) + \ln \Gamma(f) \quad (4.6)$$

where constant C and $\chi(f)$ are defined in Eq. (1.3) and Eq. (1.4), respectively, while $\psi(1)$ is the Euler constant, which is equal to 0.577216.

The general solution to Eq. (3.9), which satisfies the initial condition $S(z=0) = S_0$, can be written in the form

$$\begin{aligned} S(z) &= S_0 \int_{a-i\infty}^{a+i\infty} \frac{df}{2\pi i} \frac{X'_f(f)}{X(f)} \exp(X(f) + fz) \\ &= S_0 \int_{a-i\infty}^{a+i\infty} \frac{df}{2\pi i} \frac{Cf - \chi(f)}{X(f)} \exp(X(f) + fz) \end{aligned} \quad (4.7)$$

where a in the contour of integration is situated to the right of all singularities of the integrand [10].

At $z=0$ Eq. (4.7) leads to

$$S(z=0) = S_0 \frac{1}{2\pi i} \int \frac{dX}{X} e^X = S_0 \quad (4.8)$$

One can see two important properties of the integrand in Eq. (4.7): (i) there are no singularities in it for $f > 0$, since the singularities of $\Gamma(1-f)$ in the exponent cancel the singularities of $X'(f)$ in the numerator of Eq. (4.7); and (ii) contribution at the pole $f \rightarrow 0$ vanishes since the numerator is equal to zero in this point. This fact arises due to the cancellation of single pole and double pole contributions. This observation means that we correctly evaluated the integral of Eq. (4.3). There is a problem in that we can justify the second term in the r.h.s. of Eq. (3.9) only for $|\vec{r} - \vec{r}'| \approx 1/Q_s \ll r$. However, at $f \rightarrow 0$ the region of $|\vec{r} - \vec{r}'| \sim r$ also contributes since $\ln(r'^2/r^2) \approx 1/f$. Strictly speaking we needed to subtract term $1/f$ from Eq. (4.3). Our observation means that it was correct to keep this term in the evaluation of the large z behaviour of $S(z)$.

The integral in Eq. (4.7) has two sources for the asymptotic behaviour: the saddle point at large f and the pole (singularities) contributions at $f \rightarrow -n$ where $n = 0, 1, 2, \dots$

To calculate the saddle point contribution at large values of f we replace all Gamma functions in Eq. (4.7) by the asymptotic expression. Therefore, the expression for $X(f)$ has the form

$$X(f) \rightarrow C \frac{f^2}{2} + 2f \ln(f) \quad \text{at } f \gg 1 \quad (4.9)$$

The equation for the saddle point is

$$C f_{SP}^{(+)} + 2 \ln(f_{SP}^{(+)}) + z = 0 \quad (4.10)$$

and

$$f_{SP}^{(+)} = -\frac{z}{C} - 2 \ln(z) \quad (4.11)$$

Therefore, the saddle point contribution has the form

$$S^+(z) = S_0 \sqrt{\frac{8\pi C}{z}} \exp\left(-\frac{1}{2} C \left(\frac{z}{C} + 2 \ln(z)\right)^2\right) \quad (4.12)$$

which is the same as our solution of Eq. (3.17), but obtained with better accuracy.

Near the singularity $f = -n$ the integrand is

$$\frac{X'}{X} e^{X+fz} = \frac{(-1)^{n-1}}{(n-1)!} \frac{1}{\ln(n+f)} \frac{1}{(n+f)^2} e^{fz} = \frac{(-1)^{n-1}}{(n-1)!} \int_0^\infty dt (n+f)^{-2+t} e^{fz} \quad (4.13)$$

Closing the contour about the point $j = -n$, we see that the contribution is equal to

$$\begin{aligned} \int_0^\infty dt (n+f)^{-2+t} e^{fz} &= e^{-nz} \int_0^\infty dt \frac{1}{\Gamma(2-t)} z^{1-t} \\ &= z \frac{1}{\ln(z)} e^{-nz} \end{aligned} \quad (4.14)$$

We used here the integral representation of the Euler gamma-function (see Ref. [19] **8.314**).

One can see from Eq. (4.14), that only the first singularity is essential since all others are suppressed at large values of z .

However, they give more than the saddle point, and because of this, we have to sum all these contributions if we want to take into account the saddle point contribution as well. The sum gives

$$S_{\Sigma}^{-}(z) = \frac{z}{\ln(z)} \left(1 - \exp\left(-e^{-z}\right) \right) \quad (4.15)$$

Finally the behaviour of the solution in the deep saturation region can be written as sum

$$S_{asymp}(z) = S_{\Sigma}^{-}(z) + S^{+}(z) \quad (4.16)$$

At large z Eq. (4.16) leads to

$$S_{asymp}(z) \rightarrow S_0 \frac{z}{\ln(z)} e^{-z} \quad (4.17)$$

This solution in the form of Eq. (4.16) has a very clear physical meaning. Each term is the suppression of gluon emission. Indeed, approaching the black disc limit, we expect that an emitted gluon could not survive, except at the impact parameters close to the edge of the hadron disc [16, 17, 18]. Inside of the disc, the dipole could elastically rescatter but cannot emit gluons. The S^{+} term in Eq. (4.16) describes the probability of dipole interactions without emission of a gluon in the leading twist. All other terms correspond to such a probability but for higher twists contributions in the BFKL equation, since the poles at $j = -n$ at $n > 1$ correspond to the contribution of higher twists to the BFKL Pomeron ³ (see for example Ref. [20]).

5 Numerical check of the new solution

Despite the analytical calculations that led us to the solution in the form of Eq. (4.17) it is necessary to check that the homogeneous equation (our master equation with zero l.h.s.) has a solution. The main reason for such a check, is that the analytical consideration is correct only for large $z \gg 1$, and the matching between negative z and large but positive z is still out of theoretical control. The second motivation for searching for a numerical solution is the result of the numerical simulation by Salam [24] who confirmed the solution of Ref. [3] rather than the solution of Eq. (4.17).

We search for the solution of the following equation

$$\frac{\partial N(y, q)}{\partial y} = \frac{\alpha_S \cdot N_C}{\pi} \left(\frac{1}{\pi} \int \frac{d^2 q'}{(q - q')^2} \left[N(y, q') - \frac{q^2}{q'^2 + (q - q')^2} N(y, q) \right] - N^2(y, q) \right) \quad (5.1)$$

³One of us (E.L.) thanks J. Bartels for very useful discussions on higher twist contributions to the BFKL equation, that he had with him a number of years ago.

This equation is the Balitsky-Kovchegov equation but in the momentum representation, where q is a conjugated variable to r while Q is a conjugated variable to b . For simplicity in Eq. (5.1) we put $Q = 0$. After integration over angles Eq. (5.1) reduces to the form

$$\frac{\partial N(y, q)}{\partial y} = \frac{\alpha_S \cdot N_C}{\pi} \left\{ \int_0^\infty dq'^2 \left[\frac{N(y, q'^2)}{|q'^2 - q^2|} - \frac{q^2}{q'^2} \cdot \frac{N(y, q^2)}{|q'^2 - q^2|} + \frac{q^2}{q'^2} \cdot \frac{N(y, q^2)}{\sqrt{4q'^4 + q^4}} \right] - N^2(y, q^2) \right\} \quad (5.2)$$

We wish to solve this equation assuming the geometric scaling behaviour of the solution, namely, this solution is a function of the only one variable

$$\hat{z} = \ln \left(Q_s^2(y, b)/k^2 \right) = \bar{\alpha}_S \frac{\chi(\gamma_{cr})}{1 - \gamma_{cr}} (Y - Y_0) - \xi - \beta(b); \quad (5.3)$$

We can express Eq. (5.2) in terms of single variable z using the following expressions:

1. $q^2 = e^{-z}$ and $dq^2 = -e^{-z} dz$
2. $\frac{\partial}{\partial y} = \bar{\alpha}_S \frac{\chi(\gamma_{cr})}{1 - \gamma_{cr}} \frac{\partial}{\partial z}$

Thus, finally we get following non-linear equation:

$$\frac{\partial N(z)}{\partial z} = \frac{1 - \gamma_{cr}}{\chi(\gamma_{cr})} \left\{ \int_{-\infty}^\infty dz' \left[\frac{N(z') \cdot e^{-z'}}{|e^{-z'} - e^{-z}|} - \frac{N(z) \cdot e^{-z}}{|e^{-z'} - e^{-z}|} + \frac{N(z) \cdot e^{-z}}{\sqrt{(2e^{-z'})^2 + (e^{-z})^2}} \right] - N^2(z) \right\} \quad (5.4)$$

As we have mentioned Eq. (2.9), we can find the scattering amplitude $N(z)$ using function $\phi(z)$. We are going to demonstrate numerically, that $\phi(z) = -(z - \log z)$ (Eq. (3.18)) minimizes the right hand side of Eq. (5.4), i.e. it can be considered as solution for homogeneous equation:

$$0 = \frac{1 - \gamma_{cr}}{\chi(\gamma_{cr})} \left\{ \int_{-\infty}^\infty dz' \left[\frac{N(z') \cdot e^{-z'}}{|e^{-z'} - e^{-z}|} - \frac{N(z) \cdot e^{-z}}{|e^{-z'} - e^{-z}|} + \frac{N(z) \cdot e^{-z}}{\sqrt{(2e^{-z'})^2 + (e^{-z})^2}} \right] - N^2(z) \right\} \quad (5.5)$$

In order to perform required calculations we have to expand Eq. (3.18) to the negative values of z . It is well known [11, 12, 13] that at the negative z , the scattering amplitude behaves as $N(z) = N_0 e^{(1 - \gamma_{cr})z}$. We also require continuation of $N(z)$ and $N'(z)$ at $z = 0$.

We also require the matching of the two solutions at $z = 0$ namely, the value of the amplitude $N(z)$ and its derivative $N'(z)$ should be equal at $z = 0$.

$$N(z) = \begin{cases} N_0 e^{(1 - \gamma_{cr})z} & z < 0; \\ N_0 + \frac{1}{2} \int_0^z \left(1 - \beta \cdot \exp \left[- (z' - \log [z' + \frac{1 - 2N_0(1 - \gamma_{cr})}{\beta}]) \right] \right) dz' & z > 0; \end{cases} \quad (5.6)$$

Parameter β , actually, accumulates all information which is beyond the precision of our solution. As we can see from Fig. 2-a variation of this parameter changes the value of the r.h.s. of the equation (see Fig. 2) that is calculated using Eq. (5.6). One can see that the r.h.s. of the equation vanishes if we choose a value of β . It means that $\phi = z - \ln z$ really is a solution to the homogeneous equation which fulfills the correct boundary condition.

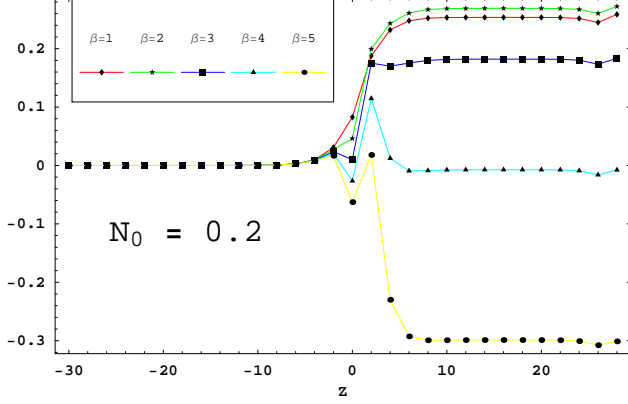


Fig. 2-a

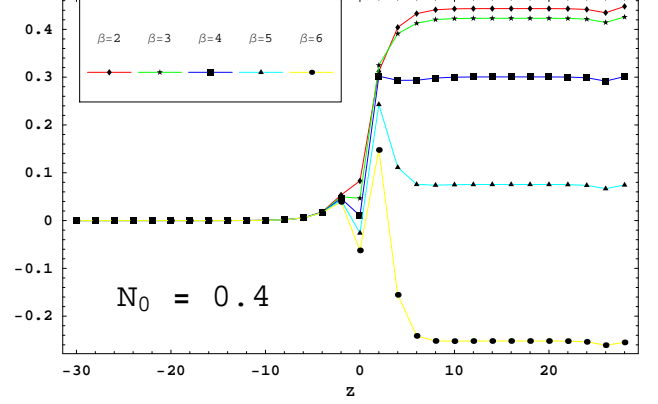


Fig. 2-b

Figure 2: In this plot we demonstrate results of numerical calculations of evaluation Eq. (5.5) for different values of β with initial condition: $N(z=0) = 0.2$ (Fig. 2-a) and $N(z=0) = 0.4$ (Fig. 2-b).

We hope that this numerical calculation will dissipate the lingering doubts that the homogeneous equation has a solution, which approaches the asymptotic solution for high energy amplitudes.

6 Beyond the Balitsky-Kovchegov equation

In Ref. [12] a new approach beyond the mean field approximation is proposed, based on the statistical interpretation of the high energy behaviour of the scattering amplitude. The main result of this paper can be understood in the following way.

The scattering amplitude for dipole-target interaction can be viewed as the average of the solution to the Balitsky-Kovchegov equation with the saturation scale Q_s , over the saturation scale. The average saturation scale is given by the equation [21]

$$\frac{d \ln \bar{Q}_s^2}{dY} = \bar{\alpha}_S \frac{\chi(\gamma_{cr})}{1 - \gamma_{cr}} - \frac{\pi^2 \bar{\alpha}_S}{2} \frac{(1 - \gamma_{cr}) \chi''(\gamma_{cr})}{\ln^2(1/\alpha_S^2)} \quad (6.1)$$

and averaging should be performed with a Gaussian weight of variance

$$\sigma^2 = c \bar{\alpha}_S Y / \ln^3(1/\alpha_S) \quad (6.2)$$

Therefore the dipole amplitude is equal to [21]

$$A(Y, r) = \frac{1}{\sqrt{2\pi}\sigma} \int d\rho N(\bar{z}, \rho) e^{-\frac{\rho^2}{2\sigma^2}} \quad (6.3)$$

where $\bar{z} = \ln(r^2 \bar{Q}_s^2)$ and $\rho = \ln(Q_s^2/\bar{Q}_s^2)$.

This averaging drastically changes the asymptotic behaviour of the first solution $N \propto e^{-\frac{z^2}{2c}}$. It transforms into

$$N \propto e^{-\frac{z^2}{2c}} \rightarrow A(\bar{z}, Y) \propto e^{-\frac{\bar{z}}{2\sigma^2}} \quad (6.4)$$

Using Eq. (6.2) one can see that Eq. (6.4) leads to the following asymptotic behaviour at fixed $r^2 < 1/\bar{Q}_s^2$

$$A(\bar{z}, Y) \propto \exp\left(-\frac{\bar{\alpha}_S}{2c} C^2 \ln^3(1/as^3) Y\right) \quad (6.5)$$

On the contrary averaging does not dramatically change Eq. (4.17) leading mostly to redefinition of the saturation scale, namely, the new scaling variable is equal to

$$\xi = \bar{z} - \frac{1}{2}\sigma^2 \quad (6.6)$$

or, in other words, instead of saturation scale \bar{Q}_s given by Eq. (6.1) we have to introduce a saturation scale

$$\tilde{Q}_s^2 = \bar{Q}_s^2 e^{-\frac{1}{2}\sigma^2} = \bar{Q}_s^2 \exp\left(-\frac{1}{2}c \frac{\bar{\alpha}_S}{\ln^3(1/\alpha_S)} Y\right) \quad (6.7)$$

In new variable the asymptotic behaviour of the scattering amplitude looks as

$$A(Y, \xi) = \left(\xi + \frac{3}{2}\sigma^2 - \frac{\sigma}{\sqrt{2\pi}}\right) e^{-\xi} = \xi e^{-\xi} \quad (6.8)$$

Therefore, this solution shows the geometrical scaling behaviour while the first one displays no such scaling behaviour.

7 Summary

We found that the Balitsky-Kovchegov homogeneous equation (see Eq. (2.1) with the l.h.s. is equal to zero) has a solution. This solution is determined by the singularities of the BFKL kernel in the anomalous dimension (Mellin conjugated variable to $\ln(r^2)$ where r is the dipole size). This solution has the following form

$$N(r^2) = 1 - C \exp(-z + \ln z) \quad (7.1)$$

where $z = \ln(r^2/R^2)$ where R is the arbitrary scale. C is an arbitrary constant that we cannot calculate. Setting $r^2 = 1/Q_s^2(Y)$ (Q_s is the saturation scale) on physical grounds we obtain that the solution to the Balitsky-Kovchegov equation has the form:

$$N(r^2) = 1 - Const \exp\left(-\ln(r^2 Q_s^2) + \ln \ln(r^2 Q_s^2)\right) \quad (7.2)$$

where *Const* cannot be determined within our accuracy.

Since it depends on the singularities of the BFKL kernel it is very dangerous to make any simplification of this kernel. Indeed, in two approximate expressions for the BFKL kernel that have been suggested (see Refs. [3, 13]) the solution of Eq. (7.2) is missed. Formally speaking, this solution originates from the singularities of the BFKL kernel. In practical terms, it means that we should be very careful approximating the full Balitsky-Kovchegov equation, by the simplified model of the BFKL kernel, namely, by diffusion approximation, in which this non-linear equation belongs to the Fisher-Kolmogorov-Petrovsky-Piscounov -type of equation (see Ref. [13]). This issue needs further careful investigation.

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